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Bicoherent states and path integrals for systems with first-class constraints

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Abstract. We study the path integral for a model with a finite number of degrees of freedom and two first-class constraints. To account for the constraints, we construct the appropriate projection operator, and, rather than the resolution of unity, use it at every time slice in the building of the coherent-state path-integral representation of the propagator. The derivation of the projection operator leads to the introduction of bicoherent states and is built by integration over properly-weighted, independent coherent-state bras and kets. The construction of the propagator using bicoherent states leads to a phase space action, which, in general, is complex and has twice as many labels as there are in the standard classical phase space action. The imaginary part of the complex action reduces to a surface term on the classical trajectories. We argue that the projection operator leads to the correct measure in the path-integral representation of the propagator. The measure, which is path dependent, is 'modulated' by the imaginary part of the action.

1. Introduction

Problems in classical physics are often conveniently described in terms of more variables than are required and such problems are said to have constraints. Mechanical systems with constraints have been studied systematically by Dirac [1]. Constraints are naturally classified as either first class or second class. First-class constraints are those whose Poisson brackets with all other constraints vanish on the constraint surface and constraints that are not first class are second class. In efforts to quantize systems with first-class constraints, variables referred to as gauge degrees of freedom become a nuisance. In the literature, there exist several widely accepted methods for path-integral quantization of systems with purely gauge degrees of freedom [2–6]. Although these schemes for quantization are varied, they all allow rather unrestricted canonical transformations within a formal phase space path integral, the validity of which often cannot be substantiated [7].

In this paper, we consider a toy model which has been studied before [8, 9]. The model has a primary and a secondary constraint, both first class. We quantize the model and, following Dirac, identify the physical subspace [1] . Then, starting with an orthonormal basis in this subspace we build a projection operator which we use in the construction of a coherent-state path-integral representation of the propagator for our model. The projection operator, which leads to the introduction of bicoherent states, gives us the correct measure for the path integral and also gives us the desired classical limit.

The paper is organized as follows. In section 2, we introduce the classical and quantum description of the model and identify its physical subspace. In section 3, which contains the bulk of the formalism developed here, we construct the projection operator and use

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it to evaluate the propagator for the case of the quadratic potential. We then obtain the classical limit from this propagator. In section 4, we study the propagator for the quartic potential. Section 5 discusses the measure obtained for the path-integral representation of the propagator. The appendix summarizes the main features of path integrals constructed using bicoherent states.

2. The toy model

We consider the dynamical system described by the Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}, y, \dot{y}) = \tfrac{1}{2}(\dot{\boldsymbol{x}} - yT\boldsymbol{x})^2 - V(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2)$ a two-dimensional vector, and y are dynamical variables. Also, $T = \mathrm{i}\tau_2$ is a 2×2 matrix where τ_2 is a Pauli matrix. As a first step toward quantization we change to the Hamiltonian formalism. The canonically conjugate momenta to the coordinates are

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \dot{\boldsymbol{x}} - yT\boldsymbol{x} \qquad \pi = \frac{\partial L}{\partial \dot{y}} = 0 \tag{2}$$

and the canonical Hamiltonian is

$$H = \tfrac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x}) + y\boldsymbol{p}T\boldsymbol{x}. \tag{3}$$

Thus, we have a mechanical system with one primary constraint $\pi = 0$. We want our primary constraint to hold at all times, so we require

$$\dot{\pi} = \{\pi, H\} = -\boldsymbol{p}T\boldsymbol{x} = -\sigma = 0 \tag{4}$$

i.e. we have a secondary constraint, $\sigma = 0$. We note that $\sigma = (p_1x_2 - p_2x_1)$ is just the generator of rotations in two dimensions. Both constraints in our problem are first class since $\{\sigma, \pi\} = 0$. Also, there are no further constraints in the problem because $\dot{\sigma} = \{\sigma, H\} = 0$. Thus, our model which has two first-class constraints has only one physical degree of freedom which can be identified as follows. Performing the canonical transformation $(\boldsymbol{x}, \boldsymbol{p}) \to (r, \theta, p_r, p_\theta)$, where (r, θ) are polar coordinates and (p_r, p_θ) are momenta conjugate to them respectively, we find that (r, p_r) are gauge invariant and can be taken as the physical variables.

In the case in which one is interested in the most general physically permissible motion, one should allow for an arbitrary gauge transformation to be performed while the system is dynamically evolving in time. Hence, we add to our Hamiltonian the two first-class constraints multiplied by their corresponding Lagrange multipliers and obtain the extended Hamiltonian, i.e.

$$H = \tfrac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x}) + y\boldsymbol{p}T\boldsymbol{x} + u\boldsymbol{p}T\boldsymbol{x} + v\pi \tag{5}$$

where u and v are Lagrange multipliers [10]. We use the extended Hamiltonian in the construction of the propagator and, henceforth, will refer to it as just the Hamiltonian unless otherwise specified.

2.1. Quantum theory

The transition to the quantum description of the system is made by promoting the dynamical variables $\boldsymbol{x}$, $\boldsymbol{p}$, y, π and the Lagrange multipliers u, v to operators. We will use the same symbol to represent the classical variables and their corresponding quantum operators since the quantity being referred to will be clear from the context. At present, we consider the harmonic oscillator potential $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^2$. So, our Hamiltonian is

$$H = \tfrac{1}{2}\boldsymbol{p}^2 + \tfrac{1}{2}\boldsymbol{x}^2 + y\boldsymbol{p}T\boldsymbol{x} + u\boldsymbol{p}T\boldsymbol{x} + v\pi. \tag{6}$$

It is useful to express our quantum mechanical problem in a second-quantized representation. Thus, using creation and annihilation operators, we define

$$\begin{aligned} x_j &= \frac{(a_j + a_j^\dagger)}{\sqrt{2}} \qquad p_j = \frac{(a_j - a_j^\dagger)}{\sqrt{2}\mathrm{i}} \qquad j = 1, 2 \\ y &= \frac{(a_3 + a_3^\dagger)}{\sqrt{2}} \qquad \pi = \frac{(a_3 - a_3^\dagger)}{\sqrt{2}\mathrm{i}} \qquad u = \frac{(a_4 + a_4^\dagger)}{\sqrt{2}} \qquad v = \frac{(a_5 + a_5^\dagger)}{\sqrt{2}} \end{aligned} \tag{7}$$

and adopt the normal-ordered Hamiltonian $\mathcal{H} =: H :$ given by

$$\begin{aligned} \mathcal{H} = a_1^\dagger a_1 + a_2^\dagger a_2 + \mathrm{i}\frac{(a_3^\dagger + a_3)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) \\ + \mathrm{i}\frac{(a_4^\dagger + a_4)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) + \frac{(a_5^\dagger + a_5)}{\sqrt{2}}\frac{(a_3 - a_3^\dagger)}{\sqrt{2}\mathrm{i}} \end{aligned} \tag{8}$$

where $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$ and $i, j = 1, 2, 3, 4, 5$. An orthonormal basis for the Hilbert space under consideration is given by the oscillator occupation number states

$$\frac{(a_1^\dagger)^l}{\sqrt{l!}}\frac{(a_2^\dagger)^m}{\sqrt{m!}}\frac{(a_3^\dagger)^n}{\sqrt{n!}}\frac{(a_4^\dagger)^r}{\sqrt{r!}}\frac{(a_5^\dagger)^s}{\sqrt{s!}}|0\rangle \qquad l, m, n, r, s = 0, 1, 2, \ldots \tag{9}$$

where the vacuum state $|0\rangle$ is defined by $a_i|0\rangle = 0$.

Given the quantization prescription and the Hilbert space indicated above, we would now like to identify the physical subspace which respects the constraints as quantum operators. First, consider the subspace in which the operators $(\boldsymbol{x}, \boldsymbol{p})$ live, which in the occupation number representation corresponds to the space spanned by the vectors

$$\frac{(a_1^\dagger)^l}{\sqrt{l!}}\frac{(a_2^\dagger)^m}{\sqrt{m!}}|0\rangle = |l, m\rangle. \tag{10}$$

In this subspace, the physical states are singled out by the condition

$$\sigma|\phi\rangle = \boldsymbol{p}T\boldsymbol{x}|\phi\rangle = \mathrm{i}(a_1^\dagger a_2 - a_2^\dagger a_1)|\phi\rangle = 0. \tag{11}$$

Thus, independent physical vectors in this subspace are obtained by applying to the vacuum state $|0\rangle$ polynomials in $a_1^\dagger$ and $a_2^\dagger$ that commute with $\sigma = \mathrm{i}(a_1^\dagger a_2 - a_2^\dagger a_1)$. The only such independent invariant polynomial is $(a_1^{\dagger 2} + a_2^{\dagger 2})$, i.e.

$$[\sigma, (a_1^{\dagger 2} + a_2^{\dagger 2})] = 0. \tag{12}$$

Hence, an orthonormal basis in the physical subspace under consideration is given by [9]

$$|\phi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!}|0\rangle \qquad k = 0, 1, 2, \ldots. \tag{13}$$

Notice that these states are energy eigenstates for the canonical Hamiltonian in (3) for a quadratic potential with eigenvalues 2, 4, 6, ... in appropriate units.

Second, the physical states in the subspace where (y, π) operate is similarly determined by the condition $\pi|\phi\rangle = 0$. In the Fock space representation we replace this by the weaker requirement $\pi^{(-)}|\phi\rangle = 0$, where $\pi^{(-)}$ is the annihilation part of π. Thus, in this subspace the physical state is just the vacuum state $|0\rangle$. Hence, our physical states invariant under the two gauge transformations are

$$|\psi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!} |0, 0, 0\rangle. \tag{14}$$

These states form an orthonormal basis for the physical space which is a subspace of the Hilbert space spanned by the vectors in (9).

3. The path integral

The principal object represented by the path integral is the propagator. Before proceeding to the construction of the propagator for our toy model with two first-class constraints, we will briefly recall the construction of the propagator in the canonical coherent-state representation for systems with a single degree of freedom and without constraints. In the canonical coherent-state representation the propagator is given by

$$\begin{aligned}\langle z''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|z'\rangle &= \int \ldots \int \langle z''|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|z_N\rangle \ldots \langle z_1|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|z'\rangle \prod_{n=1}^{N} \frac{\mathrm{d}^2 z_n}{\pi} \\ &= \int \exp\left[\mathrm{i}\int_0^T [\tfrac{1}{2}(p\dot{q} - q\dot{p}) - H(p, q)]\,\mathrm{d}t\right]\mathcal{D}p\mathcal{D}q \end{aligned} \tag{15}$$

where $H(p, q) = \langle p, q|\mathcal{H}|p, q\rangle$. The states $|z\rangle = |p, q\rangle$ are canonical coherent states and are given by

$$|z\rangle = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{16}$$

where $z = (q + \mathrm{i}p)/\sqrt{2}$. Also, the state $|n\rangle$ is the nth excited harmonic oscillator eigenstate. In (15) the resolution of unity

$$1 = \sum_{n=0}^{\infty} |n\rangle\langle n| = \int |z\rangle\langle z| \frac{\mathrm{d}^2 z}{\pi} \tag{17}$$

has been used at every time slice during the construction of the path integral [11].

The principal premise of this paper is that in the construction of the path integral representation of the propagator for a constrained system, rather than the resolution of unity, one should use a projection operator at every time slice. The projection operator ensures that at every infinitesimal time step forward, the evolving state is projected onto the physical subspace.

3.1. The projection operator

We shall now construct the appropriate projection operator. In the space spanned by the basis vectors

$$|l, m, n\rangle = \frac{(a_1^\dagger)^l (a_2^\dagger)^m (a_3^\dagger)^n}{\sqrt{l!m!n!}} |0, 0, 0\rangle \tag{18}$$

the physical subspace is spanned by the orthonormal vectors

$$|\psi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!} |0, 0, 0\rangle \tag{19}$$

as already noted. Hence, a projection operator which will project vectors onto the physical subspace is

$$P' = \sum_{k=0}^{\infty} |\psi_k\rangle\langle\psi_k|. \tag{20}$$

In order to use this projection operator in a path integral, we will write it as an integral in a fashion similar to how the unit operator is written as $1 = \int |z\rangle\langle z|\, \mathrm{d}^2z/\pi$. To this end we note that

$$a_1^l a_2^m |\gamma, \delta\rangle = \gamma^l \delta^m |\gamma, \delta\rangle \qquad \langle\alpha, \beta| a_1^{\dagger l} a_2^{\dagger m} = \langle\alpha, \beta|\alpha^{*l}\beta^{*m} \tag{21}$$

where $|\alpha, \beta\rangle$ and $|\gamma, \delta\rangle$ are canonical coherent states. We now show how to write our projection operator in an integral representation:

$$\begin{aligned} P' &= \sum_{k=0}^{\infty} |\psi_k\rangle\langle\psi_k| \\ &= \sum_{k=0}^{\infty} \left\{ \int \frac{\mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \mathrm{d}^2\eta}{\pi^3} |\alpha, \beta, \eta\rangle\langle\alpha, \beta, \eta| \right\} |\psi_k\rangle\langle\psi_k| \left\{ \int \frac{\mathrm{d}^2\gamma\, \mathrm{d}^2\delta\, \mathrm{d}^2\xi}{\pi^3} |\gamma, \delta, \xi\rangle\langle\gamma, \delta, \xi| \right\} \end{aligned} \tag{22}$$

where in the expression above we have multiplied the projection operator by unity on either side and so we have

$$\begin{aligned} P' &= \sum_{k=0}^{\infty} \int \frac{\mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \mathrm{d}^2\eta\, \mathrm{d}^2\gamma\, \mathrm{d}^2\delta\, \mathrm{d}^2\xi}{\pi^6} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi|\langle\alpha, \beta, \eta|\psi_k\rangle\langle\psi_k|\gamma, \delta, \xi\rangle \\ &= \sum_{k=0}^{\infty} \int \frac{\mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \mathrm{d}^2\eta\, \mathrm{d}^2\gamma\, \mathrm{d}^2\delta\, \mathrm{d}^2\xi}{\pi^6} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \frac{(\alpha^{*2} + \beta^{*2})^k}{2^k k!} \frac{(\gamma^2 + \delta^2)^k}{2^k k!} \\ &\qquad \times \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2)]. \end{aligned} \tag{23}$$

In arriving at (23) we have used the fact that $\langle z|0\rangle = \langle 0|z\rangle = \exp(-\frac{1}{2}|z|^2)$. Next, using the Kronecker delta function in the form $\delta_{kl} = \int \exp\{\mathrm{i}(k-l)\theta\}\, \mathrm{d}\theta/2\pi$, we can write our projection operator as

$$\begin{aligned} P' = \sum_{k,l=0}^{\infty} \int & \frac{\mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \mathrm{d}^2\eta\, \mathrm{d}^2\gamma\, \mathrm{d}^2\delta\, \mathrm{d}^2\xi}{\pi^6} \frac{\mathrm{d}\theta}{2\pi} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \frac{(\alpha^{*2} + \beta^{*2})^k}{2^k k!} \frac{(\gamma^2 + \delta^2)^l}{2^l l!} \mathrm{e}^{\mathrm{i}(k-l)\theta} \\ & \times \mathrm{e}^{-1/2(|\alpha|^2+|\beta|^2+|\eta|^2+|\gamma|^2+|\delta|^2+|\xi|^2)} \end{aligned} \tag{24}$$

and we notice that the summation above can be converted to an exponential. Finally, we obtain for our projection operator

$$\begin{aligned} P' = \int & \frac{\mathrm{d}^2\alpha\, \mathrm{d}^2\beta\, \mathrm{d}^2\eta\, \mathrm{d}^2\gamma\, \mathrm{d}^2\delta\, \mathrm{d}^2\xi}{\pi^6} \frac{\mathrm{d}\theta}{2\pi} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \\ & \times \exp\bigg[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2) \\ & + \frac{(\alpha^{*2} + \beta^{*2})}{2} \mathrm{e}^{\mathrm{i}\theta} + \frac{(\gamma^2 + \delta^2)}{2} \mathrm{e}^{-\mathrm{i}\theta} \bigg]. \end{aligned} \tag{25}$$

The form of the projection operator in the above equation suggests the name *bicoherent states*, the term bicoherent alluding to the fact that the projection operator is a weighted integral over independent coherent-state bras and kets.

Going back to the extended Hamiltonian in (6), recall that all primary and secondary constraints, in the language of Dirac, appear in the Hamiltonian accompanied by their respective Lagrange multipliers and in the quantization process the Lagrange multipliers are also promoted to operators. Thus, to account for the Lagrange multipliers our projection operator becomes

$$P = \int \frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,\mathrm{d}^2\eta\,\mathrm{d}^2\gamma\,\mathrm{d}^2\delta\,\mathrm{d}^2\xi\,\mathrm{d}^2\rho\,\mathrm{d}^2\sigma}{\pi^8}\frac{\mathrm{d}\theta}{2\pi}|\alpha,\beta,\eta,\rho,\sigma\rangle\langle\gamma,\delta,\xi,\rho,\sigma| \times \exp\bigg[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2+|\eta|^2+|\gamma|^2+|\delta|^2+|\xi|^2)+\frac{(\alpha^{*2}+\beta^{*2})}{2} \times \mathrm{e}^{\mathrm{i}\theta}+\frac{(\gamma^2+\delta^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\bigg] \tag{26}$$

where in the expression above $a_4|\alpha,\beta,\eta,\rho,\sigma\rangle = \rho|\alpha,\beta,\eta,\rho,\sigma\rangle$ and $a_5|\alpha,\beta,\eta,\rho,\sigma\rangle = \sigma|\alpha,\beta,\eta,\rho,\sigma\rangle$. One can easily verify that the operator P satisfies the two defining properties of a projection operator, namely $P^\dagger = P$ and $P^2 = P$.

We will use the projection operator in (26) in the construction of the path-integral representation of the propagator in the next subsection. At this point, however, the interested reader may want to digress to the appendix where we discuss the salient features of path integrals constructed using bicoherent states.

3.2. The propagator

We are now equipped with the necessary tools to derive the path-integral representation of the propagator. We will calculate the matrix element of the evolution operator between canonical coherent states, i.e. $\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle$. According to our premise, the propagator is

$$\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle = \lim_{N\to\infty}\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}P_N\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}P_{N-1}\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}} \ldots P_1\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle \tag{27}$$

where at each time slice we have inserted

$$P_n = \int \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n\,\mathrm{d}^2\eta_n\,\mathrm{d}^2\gamma_n\,\mathrm{d}^2\delta_n\,\mathrm{d}^2\xi_n\,\mathrm{d}^2\rho_n\,\mathrm{d}^2\sigma_n}{\pi^8}\frac{\mathrm{d}\theta_n}{2\pi}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle\langle\gamma_n,\delta_n,\xi_n,\rho_n,\sigma_n| \times \exp\bigg[-\tfrac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2) +\frac{(\alpha_n^{*2}+\beta_n^{*2})}{2}\,\mathrm{e}^{\mathrm{i}\theta_n}+\frac{(\gamma_n^2+\delta_n^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta_n}\bigg] \tag{28}$$

the projection operator in (26) with $(N+1)\epsilon = T$ and $n = 1, 2, \ldots, N$. Although for many problems one application of the projection operator would be adequate, the projection operator is used at every time slice to account for models in which the Hamiltonian and the time evolution are not gauge invariant. For such models a physical state would be mapped, by the Schrödinger equation, onto a state that fails to be gauge invariant. Thus, the propagator is

$$\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle = \lim_{N\to\infty}\int\ldots\int\prod_{n=0}^{N}\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle \times\prod_{n=1}^{N}\mathrm{d}\mu_n'' \tag{29}$$

with the measure at each time slice being

$$\mathrm{d}\mu_n'' = \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n\,\mathrm{d}^2\eta_n\,\mathrm{d}^2\gamma_n\,\mathrm{d}^2\delta_n\,\mathrm{d}^2\xi_n\,\mathrm{d}^2\rho_n\,\mathrm{d}^2\sigma_n}{\pi^8}\frac{\mathrm{d}\theta_n}{2\pi} \times \exp\Bigg[-\tfrac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2) + \frac{(\alpha_n^{*2}+\beta_n^{*2})}{2}\,\mathrm{e}^{\mathrm{i}\theta_n} + \frac{(\gamma_n^2+\delta_n^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta_n}\Bigg] \tag{30}$$

and the boundary conditions given by

$$\begin{aligned}&(\gamma_{N+1},\delta_{N+1},\xi_{N+1},\rho_{N+1},\sigma_{N+1}) = (\alpha'',\beta'',\eta'',\rho'',\sigma'')\\&(\alpha_0,\beta_0,\eta_0,\rho_0,\sigma_0) = (\alpha',\beta',\eta',\rho',\sigma').\end{aligned} \tag{31}$$

For small ϵ, we have, to order ϵ,

$$\begin{aligned}&\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle\\&\quad\simeq\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|[1-\mathrm{i}\epsilon\mathcal{H}]|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle\\&\quad=\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle[1-\mathrm{i}\epsilon H_{n+1,n}]\end{aligned} \tag{32}$$

where in the expression above

$$H_{n+1,n} = \frac{\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\mathcal{H}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle}{\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle}. \tag{33}$$

Thus, using the fact that $[1-\mathrm{i}\epsilon H_{n+1,n}]\simeq \mathrm{e}^{-\mathrm{i}\epsilon H_{n+1,n}}$ we are led to the following expression, provided the integrals exist, for the propagator:

$$\begin{aligned}&\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle\\&\quad=\lim_{N\to\infty}\int\ldots\int\prod_{n=0}^{N}\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle\\&\quad\quad\times\mathrm{e}^{-\mathrm{i}\epsilon H_{n+1,n}}\prod_{n=1}^{N}\mathrm{d}\mu_n''.\end{aligned} \tag{34}$$

The canonical coherent-state overlap at each time slice in the above expression is

$$\begin{aligned}&\langle\gamma_{n+1},\delta_{n+1},\xi_{n+1},\rho_{n+1},\sigma_{n+1}|\alpha_n,\beta_n,\eta_n,\rho_n,\sigma_n\rangle\\&\quad=\exp[\gamma_{n+1}^*\alpha_n+\delta_{n+1}^*\beta_n+\xi_{n+1}^*\eta_n+\rho_{n+1}^*\rho_n+\sigma_{n+1}^*\sigma_n]\\&\quad\quad\times\exp[-\tfrac{1}{2}(|\gamma_{n+1}|^2+|\delta_{n+1}|^2+|\xi_{n+1}|^2+|\rho_{n+1}|^2+|\sigma_{n+1}|^2\\&\quad\quad+|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\rho_n|^2+|\sigma_n|^2)]\end{aligned} \tag{35}$$

and we notice that the factor

$$\begin{aligned}&\exp[-\tfrac{1}{2}(|\gamma_{n+1}|^2+|\delta_{n+1}|^2+|\xi_{n+1}|^2+|\rho_{n+1}|^2+|\sigma_{n+1}|^2\\&\quad+|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\rho_n|^2+|\sigma_n|^2)]\end{aligned}$$

except at the endpoints, can be absorbed in the measure $\mathrm{d}\mu_n''$. Hence, our propagator becomes

$$\begin{aligned}&\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle\\&\quad=\lim_{N\to\infty}\exp[-\tfrac{1}{2}(|\alpha''|^2+|\beta''|^2+|\eta''|^2+|\rho''|^2+|\sigma''|^2+|\alpha'|^2\\&\quad\quad+|\beta'|^2+|\eta'|^2+|\rho'|^2+|\sigma'|^2)]\int\ldots\int\exp\Bigg[\sum_{n=0}^{N}(\gamma_{n+1}^*\alpha_n+\delta_{n+1}^*\beta_n+\xi_{n+1}^*\eta_n\\&\quad\quad+\rho_{n+1}^*\rho_n+\sigma_{n+1}^*\sigma_n-\mathrm{i}\epsilon H_{n+1,n})\Bigg]\prod_{n=1}^{N}\mathrm{d}\mu_n'\end{aligned} \tag{36}$$

where the overall factor arose from the endpoints of the term absorbed in the measure, which at each time slice has changed slightly and is now given by

$$
\begin{aligned}
\mathrm{d}\mu_n' = {} & \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n\,\mathrm{d}^2\eta_n\,\mathrm{d}^2\gamma_n\,\mathrm{d}^2\delta_n\,\mathrm{d}^2\xi_n\,\mathrm{d}^2\rho_n\,\mathrm{d}^2\sigma_n}{\pi^8}\frac{\mathrm{d}\theta_n}{2\pi} \\
& \times \exp\Bigg[-(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2+|\rho_n|^2+|\sigma_n|^2) \\
& +\frac{(\alpha_n^{*2}+\beta_n^{*2})}{2}\,\mathrm{e}^{\mathrm{i}\theta_n}+\frac{(\gamma_n^2+\delta_n^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta_n}\Bigg]. \qquad (37)
\end{aligned}
$$

Our goal is to express the right-hand side of (36) as a path integral and in preparation towards this objective we rewrite part of the exponent in the integrand of this equation as follows:

$$
\begin{aligned}
& \sum_{n=0}^{N}[\gamma_{n+1}^*\alpha_n+\delta_{n+1}^*\beta_n+\xi_{n+1}^*\eta_n+\rho_{n+1}^*\rho_n+\sigma_{n+1}^*\sigma_n] \\
& \quad = \sum_{n=0}^{N}\tfrac{1}{2}\{(\gamma_{n+1}^*-\gamma_n^*)\alpha_n-\gamma_{n+1}^*(\alpha_{n+1}-\alpha_n)+(\delta_{n+1}^*-\delta_n^*)\beta_n \\
& \quad -\delta_{n+1}^*(\beta_{n+1}-\beta_n)+(\xi_{n+1}^*-\xi_n^*)\eta_n-\xi_{n+1}^*(\eta_{n+1}-\eta_n)+(\rho_{n+1}^*-\rho_n^*)\rho_n \\
& \quad -\rho_{n+1}^*(\rho_{n+1}-\rho_n)+(\sigma_{n+1}^*-\sigma_n^*)\sigma_n-\sigma_{n+1}^*(\sigma_{n+1}-\sigma_n)\} \\
& \quad +\sum_{n=0}^{N}\tfrac{1}{2}\{\gamma_n^*\alpha_n+\gamma_{n+1}^*\alpha_{n+1}+\delta_n^*\beta_n+\delta_{n+1}^*\beta_{n+1}+\xi_n^*\eta_n+\xi_{n+1}^*\eta_{n+1} \\
& \quad +\rho_n^*\rho_n+\rho_{n+1}^*\rho_{n+1}+\sigma_n^*\sigma_n+\sigma_{n+1}^*\sigma_{n+1}\}. \qquad (38)
\end{aligned}
$$

In the equation above, the terms γ_0, δ_0, ξ_0, α_{N+1}, β_{N+1} and η_{N+1} have not yet been defined. The factors containing these terms cancel, and so these terms can take on arbitrary values and are at our disposal. We shall assign them the following values: $(\gamma_0, \delta_0, \xi_0) = (\alpha', \beta', \eta')$ and $(\alpha_{N+1}, \beta_{N+1}, \eta_{N+1}) = (\alpha'', \beta'', \eta'')$. The choice of these special values will become clear presently. Going back to (38), we notice the second term on the right-hand side of this equation can be absorbed in the measure, so our propagator can be written as

$$
\begin{aligned}
& \langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle \\
& \quad = \lim_{N\to\infty}\int\ldots\int\exp\Bigg\{\sum_{n=0}^{N}(\tfrac{1}{2}[(\gamma_{n+1}^*-\gamma_n^*)\alpha_n-\gamma_{n+1}^*(\alpha_{n+1}-\alpha_n) \\
& \quad +(\delta_{n+1}^*-\delta_n^*)\beta_n-\delta_{n+1}^*(\beta_{n+1}-\beta_n)+(\xi_{n+1}^*-\xi_n^*)\eta_n-\xi_{n+1}^*(\eta_{n+1}-\eta_n) \\
& \quad +(\rho_{n+1}^*-\rho_n^*)\rho_n-\rho_{n+1}^*(\rho_{n+1}-\rho_n)+(\sigma_{n+1}^*-\sigma_n^*)\sigma_n-\sigma_{n+1}^*(\sigma_{n+1}-\sigma_n)] \\
& \quad -\mathrm{i}\epsilon H_{n+1,n})\Bigg\}\prod_{n=1}^{N}\mathrm{d}\mu_n \qquad (39)
\end{aligned}
$$

where the measure at each time slice has changed again and is now

$$
\begin{aligned}
\mathrm{d}\mu_n = {} & \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n\,\mathrm{d}^2\eta_n\,\mathrm{d}^2\gamma_n\,\mathrm{d}^2\delta_n\,\mathrm{d}^2\xi_n\,\mathrm{d}^2\rho_n\,\mathrm{d}^2\sigma_n}{\pi^8}\frac{\mathrm{d}\theta_n}{2\pi} \\
& \times \exp\Bigg[-(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2)+\frac{(\alpha_n^{*2}+\beta_n^{*2})}{2}\,\mathrm{e}^{\mathrm{i}\theta_n} \\
& +\frac{(\gamma_n^2+\delta_n^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta_n}+\gamma_n^*\alpha_n+\delta_n^*\beta_n+\xi_n^*\eta_n\Bigg]. \qquad (40)
\end{aligned}
$$

The overall factor got cancelled by the endpoints of the new term absorbed in the measure. Now, interchanging the order of the limit and the integration in (39) we write for the propagator, in a formal way, the form it takes over continuous and differentiable paths as

$$\begin{aligned}\langle\alpha'',\beta'',\eta'',\rho'',\sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha',\beta',\eta',\rho',\sigma'\rangle\\ = \int \exp\bigg[\mathrm{i}\int_0^T \{\tfrac{\mathrm{i}}{2}(\gamma^*\dot\alpha-\dot\gamma^*\alpha+\delta^*\dot\beta-\dot\delta^*\beta+\xi^*\dot\eta-\dot\xi^*\eta\\ +\rho^*\dot\rho-\dot\rho^*\rho+\sigma^*\dot\sigma-\dot\sigma^*\sigma)-\langle\mathcal{H}\rangle\}\,\mathrm{d}t\bigg]\mathcal{D}\mu \qquad (41)\end{aligned}$$

where $\mathcal{D}\mu=\prod_n \mathrm{d}\mu_n$, and $\langle\mathcal{H}\rangle$ in the expression above is given by

$$\begin{aligned}\langle\mathcal{H}\rangle &= \frac{\langle\gamma,\delta,\xi,\rho,\sigma|\mathcal{H}|\alpha,\beta,\eta,\rho,\sigma\rangle}{\langle\gamma,\delta,\xi,\rho,\alpha|\alpha,\beta,\eta,\rho,\sigma\rangle}\\ &=\gamma^*\alpha+\delta^*\beta+\mathrm{i}\frac{(\xi^*+\eta)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)+\mathrm{i}\frac{(\rho^*+\rho)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)+\frac{(\sigma^*+\sigma)}{\sqrt2}\frac{(\eta-\xi^*)}{\sqrt2\mathrm{i}} \qquad (42)\end{aligned}$$

for the Hamiltonian $\mathcal{H}$ in (8). Hence, according to (41) our phase space action for continuous and differentiable paths is

$$\begin{aligned}S=\int_0^T\bigg\{\left(\frac{\mathrm{i}}{2}\right)[\gamma^*\dot\alpha-\dot\gamma^*\alpha+\delta^*\dot\beta-\dot\delta^*\beta+\xi^*\dot\eta-\dot\xi^*\eta+\rho^*\dot\rho-\dot\rho^*\rho+\sigma^*\dot\sigma-\dot\sigma^*\sigma]\\ -\bigg[\gamma^*\alpha+\delta^*\beta+\mathrm{i}\frac{(\xi^*+\eta)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)+\mathrm{i}\frac{(\rho^*+\rho)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)\\ +\frac{(\sigma^*+\sigma)}{\sqrt2}\frac{(\eta-\xi^*)}{\sqrt2\mathrm{i}}\bigg]\bigg\}\,\mathrm{d}t \qquad (43)\end{aligned}$$

with the following boundary conditions:

$$\begin{aligned}&(\alpha(0),\beta(0),\eta(0),\rho(0),\sigma(0))=(\gamma(0),\delta(0),\xi(0),\rho(0),\sigma(0))=(\alpha',\beta',\eta',\rho',\sigma')\\ &(\alpha(T),\beta(T),\eta(T),\rho(T),\sigma(T))=(\gamma(T),\delta(T),\xi(T),\rho(T),\sigma(T))\\ &\qquad=(\alpha'',\beta'',\eta'',\rho'',\sigma''). \qquad (44)\end{aligned}$$

3.3. The classical limit

We will now study the classical equations of motion obtained from our phase space action. However, before we do so, we add to our action a total time derivative which of course will not effect the equations of motion and write it as

$$\begin{aligned}S=\int_0^T\bigg\{\mathrm{i}[\gamma^*\dot\alpha+\delta^*\dot\beta+\xi^*\dot\eta+\rho^*\dot\rho+\sigma^*\dot\sigma]-\bigg[\gamma^*\alpha+\delta^*\beta+\mathrm{i}\frac{(\xi^*+\eta)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)\\ +\mathrm{i}\frac{(\rho^*+\rho)}{\sqrt2}(\gamma^*\beta-\delta^*\alpha)+\frac{(\sigma^*+\sigma)}{\sqrt2}\frac{(\eta-\xi^*)}{\sqrt2\mathrm{i}}\bigg]\bigg\}\,\mathrm{d}t. \qquad (45)\end{aligned}$$

Notice our action is generally complex, i.e. $S=S_1+\mathrm{i}S_2$. Varying S with respect to α, β, η, γ^*, δ^*, ξ^*, ρ^* and σ^* while keeping their endpoints fixed we get the following equations of motion:

$$\begin{aligned}
\dot{\alpha} &= \frac{\alpha}{\mathrm{i}} + \frac{(\xi^* + \eta)}{\sqrt{2}}\beta + \frac{(\rho^* + \rho)}{\sqrt{2}}\beta & \dot{\gamma} &= \frac{\gamma}{\mathrm{i}} + \frac{(\xi + \eta^*)}{\sqrt{2}}\delta + \frac{(\rho^* + \rho)}{\sqrt{2}}\delta \\
\dot{\beta} &= \frac{\beta}{\mathrm{i}} - \frac{(\xi^* + \eta)}{\sqrt{2}}\alpha - \frac{(\rho^* + \rho)}{\sqrt{2}}\alpha & \dot{\delta} &= \frac{\delta}{\mathrm{i}} - \frac{(\xi + \eta^*)}{\sqrt{2}}\gamma - \frac{(\rho^* + \rho)}{\sqrt{2}}\gamma \\
\dot{\eta} &= \frac{(\gamma^*\beta - \delta^*\alpha)}{\sqrt{2}} + \frac{(\sigma^* + \sigma)}{2} & \dot{\xi} &= \frac{(\alpha^*\delta - \beta^*\gamma)}{\sqrt{2}} + \frac{(\sigma^* + \sigma)}{2} \\
\dot{\rho} &= \frac{(\gamma^*\beta - \delta^*\alpha)}{\sqrt{2}} & \dot{\sigma} &= \frac{(\xi^* - \eta)}{2}.
\end{aligned} \tag{46}$$

Now, consider the paths $\alpha(t)$ and $\gamma(t)$; using the boundary conditions (44) and the evolution equations (46) we find that

$$\alpha(0) = \gamma(0) = \alpha' \qquad \dot{\alpha}(0) = \dot{\gamma}(0) = \frac{\alpha'}{\mathrm{i}} + \frac{(\eta'^* + \eta')}{\sqrt{2}}\beta' + \frac{(\rho'^* + \rho')}{\sqrt{2}}\beta'. \tag{47}$$

These are sufficient conditions for $\alpha(t) = \gamma(t)$, i.e. they evolve along identical paths. One can easily check that the pairs of paths $(\beta(t), \delta(t))$ and $(\eta(t), \xi(t))$ also start off with the same initial conditions and so we have $\beta(t) = \delta(t)$ and $\eta(t) = \xi(t)$. We will take up the equations for ρ and σ later; these equations determine the Lagrange multipliers.

To further our study of the classical equations let us define the complex quantities α, β, η, γ, δ, ξ, ρ and σ as follows:

$$\begin{aligned}
\alpha &= \frac{(q_1 + \mathrm{i}p_1)}{\sqrt{2}} & \beta &= \frac{(q_2 + \mathrm{i}p_2)}{\sqrt{2}} & \eta &= \frac{(q_3 + \mathrm{i}p_3)}{\sqrt{2}} & \gamma &= \frac{(q_4 + \mathrm{i}p_4)}{\sqrt{2}} \\
\delta &= \frac{(q_5 + \mathrm{i}p_5)}{\sqrt{2}} & \xi &= \frac{(q_6 + \mathrm{i}p_6)}{\sqrt{2}} & \rho &= \frac{(q_7 + \mathrm{i}p_7)}{\sqrt{2}} & \sigma &= \frac{(q_8 + \mathrm{i}p_8)}{\sqrt{2}}.
\end{aligned} \tag{48}$$

Notice that $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$. Using this fact and the definitions in (48) we write the evolution equations (46) in terms of the variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8)$ as follows:

$$\begin{aligned}
&\dot{q}_1 = p_1 + (q_3 + q_7)q_2 \qquad \dot{p}_1 = -q_1 + (q_3 + q_7)p_2 \qquad \dot{q}_2 = p_2 - (q_3 + q_7)q_1 \\
&\dot{p}_2 = -q_2 - (q_3 + q_7)p_1 \qquad \dot{q}_3 = q_8 \qquad \dot{p}_3 = (p_2 q_1 - p_1 q_2) \qquad \dot{q}_7 = 0 \\
&\dot{p}_7 = (p_2 q_1 - p_1 q_2) \qquad \dot{q}_8 = 0 \qquad \dot{p}_8 = -p_3.
\end{aligned} \tag{49}$$

In order to compare the above equations of motion with those for the Hamiltonian $H = (\boldsymbol{k}^2/2) + (\boldsymbol{x}^2/2) + y\boldsymbol{k}T\boldsymbol{x} + \lambda\boldsymbol{k}T\boldsymbol{x} + \zeta\pi$ we introduce a slightly extended classical phase space action S', which will give us classical equations of motion in one-to-one correspondence with the equations in (49). The extended action is

$$S' = \int_0^T \left\{ [k_1\dot{x}_1 + k_2\dot{x}_2 + \pi\dot{y} + p_\lambda\dot{\lambda} + p_\zeta\dot{\zeta}] - \left[\frac{\boldsymbol{k}^2}{2} + \frac{\boldsymbol{x}^2}{2} + y\boldsymbol{k}T\boldsymbol{x} + \lambda\boldsymbol{k}T\boldsymbol{x} + \zeta\pi \right] \right\} \mathrm{d}t \tag{50}$$

where (λ, ζ) are Lagrange multipliers and (p_λ, p_ζ) are their respective conjugate momenta. The evolution equations obtained from the action S' are

$$\begin{aligned}
&\dot{x}_1 = k_1 + (y + \lambda)x_2 \qquad \dot{k}_1 = -x_1 + (y + \lambda)k_2 \qquad \dot{x}_2 = k_2 - (y + \lambda)x_1 \\
&\dot{k}_2 = -x_2 - (y + \lambda)k_1 \qquad \dot{y} = \zeta \qquad \dot{\pi} = (x_1 k_2 - k_1 x_2) \qquad \dot{\lambda} = 0 \\
&\dot{p}_\lambda = (x_1 k_2 - k_1 x_2) \qquad \dot{\zeta} = 0 \qquad \dot{p}_\zeta = -\pi.
\end{aligned} \tag{51}$$

Comparing equations in (49) and (51) we see that with the identification

$$(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8) \leftrightarrow (x_1, k_1, x_2, k_2, x_3, k_3, \lambda, p_\lambda, \zeta, p_\zeta) \tag{52}$$

the two sets of equations are the same. Thus, we conclude that the action obtained from our quantum propagator gives us the desired classical evolution equations and hence the right classical limit.

We will now note two interesting features about the classical limit of the formalism developed here. First, substituting the definitions (48) in our complex action $S = S_1 + \mathrm{i}S_2$ in (45), we obtain for the real part of our action

$$\begin{aligned} S_1 = \tfrac{1}{2}\int_0^T \{&[(p_4\dot{q}_1 - q_4\dot{p}_1) + (p_5\dot{q}_2 - q_5\dot{p}_2) + (p_6\dot{q}_3 - q_6\dot{p}_3) + (p_7\dot{q}_7 - q_7\dot{p}_7) \\ &+(p_8\dot{q}_8 - q_8\dot{p}_8)] - [(q_4q_1 + p_4p_1) + (q_5q_2 + p_5p_2)] - q_8(p_3 + p_6) \\ &+\tfrac{1}{2}(p_6 - p_3)[(q_5q_1 + p_5p_1) - (q_4q_2 + p_4p_2)] \\ &+\tfrac{1}{2}(q_6 + q_3)[(p_5q_1 - q_5p_1) - (p_4q_2 - q_4p_2)] \\ &+q_7[(p_5q_1 - q_5p_1) - (p_4q_2 - q_4p_2)]\}\,\mathrm{d}t \end{aligned} \tag{53}$$

while the imaginary part of our complex action is given by

$$\begin{aligned} S_2 = \tfrac{1}{2}\int_0^T \{&[q_4\dot{q}_1 + p_4\dot{p}_1 + q_5\dot{q}_2 + p_5\dot{p}_2 + q_6\dot{q}_3 + p_6\dot{p}_3 + q_7\dot{q}_7 + p_7\dot{p}_7 + q_8\dot{q}_8 + p_8\dot{p}_8] \\ &+[(p_4q_1 - q_4p_1) + (p_5q_2 - q_5p_2)] + q_8(q_3 - q_6) \\ &-\tfrac{1}{2}(q_6 + q_3)[(q_4q_2 + p_4p_2) - (q_5q_1 + p_5p_1)] - \tfrac{1}{2}(p_6 - p_3)[(p_5q_1 - q_5p_1) \\ &-(p_4q_2 - q_4p_2)] - q_7[(q_4q_2 + p_4p_2) - (q_5q_1 + p_5p_1)]\}\,\mathrm{d}t. \end{aligned} \tag{54}$$

Extremizing S_1 and S_2 in equations (53) and (54), respectively, while keeping the endpoints of the paths in them fixed, we can obtain equations of motion for the dynamical variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_4, p_4, q_5, p_5, q_6, p_6)$. We find that for each of these variables the evolution equations obtained from S_1 is *identical* to the one obtained from S_2. The second interesting fact is the following. We saw that on the classical trajectories $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$, which is equivalently stated as $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$. Substituting this fact in (53) for S_1 we find that, up to a total derivative, S_1 on the classical trajectories reduces to

$$\begin{aligned} S_1 \to \int_0^T \{&[p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 + p_7\dot{q}_7 + p_8\dot{q}_8] - \tfrac{1}{2}(q_1^2 + p_1^2) - \tfrac{1}{2}(q_2^2 + p_2^2) \\ &-q_3(p_1q_2 - q_1p_2) - q_7(p_1q_2 - q_1p_2) - q_8p_3]\}\,\mathrm{d}t \end{aligned} \tag{55}$$

exactly the standard classical phase space action in (50). The imaginary part S_2 of the action with the above substitution becomes

$$S_2 \to \tfrac{1}{2}\int_0^T [q_1\dot{q}_1 + p_1\dot{p}_1 + q_2\dot{q}_2 + p_2\dot{p}_2 + q_3\dot{q}_3 + p_3\dot{p}_3 + q_7\dot{q}_7 + p_7\dot{p}_7 + q_8\dot{q}_8 + p_8\dot{p}_8]\,\mathrm{d}t \tag{56}$$

a surface term. So S_2 on the classical trajectories gives rise to only an overall 'phase factor' in the propagator.

3.4. *The constraint hypersurface*

Here we study the restrictions on the states over which the matrix element of the evolution operator is evaluated in the propagator. Consider the following equations from the set (49):

$$\begin{aligned} &\dot{q}_3 = q_8 \qquad \dot{p}_3 = (p_2q_1 - p_1q_2) \qquad \dot{q}_7 = 0 \\ &\dot{p}_7 = (p_2q_1 - p_1q_2) \qquad \dot{q}_8 = 0 \qquad \dot{p}_8 = -p_3. \end{aligned} \tag{57}$$

In the classical description of the model we are studying we had the two constraints $\pi = 0$ and $\boldsymbol{pTx} = (p_1x_2 - p_2x_1) = 0$. So in (57) we want $p_3 = \pi = 0$ and $\dot{p}_3 = \dot{p}_7 = 0$. Thus, the solutions to these equations are

$$q_8 = c_1 \qquad p_8 = c_2 \qquad q_7 = c_3 \qquad p_7 = c_4 \qquad q_3 = c_1 t + c_5 \qquad p_3 = 0 \tag{58}$$

where c_1, c_2, c_3, c_4 and c_5 are real constants which are determined by the particular classical solution one is interested in. Also note that $\dot{p}_3(t) = (p_2q_1 - p_1q_2) = 0$ implies, in particular, $\dot{p}_3(0) = 2(\alpha'_R\beta'_I - \alpha'_I\beta'_R) = 0$ and $\dot{p}_3(T) = 2(\alpha''_R\beta''_I - \alpha''_I\beta''_R) = 0$, where (α_R, β_R) and (α_I, β_I) are the real and imaginary parts of (α, β), respectively. These restrictions on (α', β') and (α'', β'') can be stated alternatively as

$$(\alpha'^*\beta' - \alpha'\beta'^*) = 0 \qquad (\alpha''^*\beta'' - \alpha''\beta''^*) = 0. \tag{59}$$

Thus, in our propagator the states $|\alpha', \beta', \eta', \rho', \sigma'\rangle$ and $|\alpha'', \beta'', \eta'', \rho'', \sigma''\rangle$ are not arbitrary but must be restricted, as discussed above, to ensure that the system remains on the constraint hypersurface in the classical limit. Hence, our propagator is

$$\begin{aligned}&\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle \\ &\qquad = \left\langle \alpha'', \beta'', \frac{c_1T + c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}} \right| \mathrm{e}^{-\mathrm{i}T\mathcal{H}} \left| \alpha', \beta', \frac{c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}} \right\rangle\end{aligned} \tag{60}$$

where (α', β') and (α'', β'') are restricted as noted in (59).

4. The quartic potential

For completeness we shall consider the quartic potential $V(\boldsymbol{x}) = \frac{1}{4}(\boldsymbol{x}^2)^2$ and show that we again obtain the correct classical limit by following the quantization procedure outlined in this paper. The Hamiltonian now is $H = \frac{1}{2}\boldsymbol{p}^2 + \frac{1}{4}(\boldsymbol{x}^2)^2 + y\boldsymbol{pTx} + u\boldsymbol{pTx} + v\pi$. In the second-quantized notation the normal-ordered form of our Hamiltonian is

$$\begin{aligned}\mathcal{H} =&: (-\tfrac{1}{4})[(a_1 - a_1^\dagger)^2 + (a_2 - a_2^\dagger)^2] + \tfrac{1}{16}[(a_1 + a_1^\dagger)^2 + (a_2 + a_2^\dagger)^2]^2 \\ &+\mathrm{i}\frac{(a_3 + a_3^\dagger)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) + \mathrm{i}\frac{(a_4 + a_4^\dagger)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) \\ &+ \frac{(a_5 + a_5^\dagger)}{\sqrt{2}}\frac{(a_3 - a_3^\dagger)}{\sqrt{2}\mathrm{i}} : .\end{aligned} \tag{61}$$

The propagator for canonical coherent states is again

$$\begin{aligned}&\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle \\ &\qquad = \lim_{n\to\infty}\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}P_N\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}\ldots P_1\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha', \beta', \eta'; , \rho', \sigma'\rangle\end{aligned} \tag{62}$$

where P_n is the projection operator in (26). Interchanging the order of the limit and the integration as usual we formally write the propagator as

$$\langle\alpha'', \beta'', \eta'', \rho'', \ \sigma''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle = \int \mathrm{e}^{\mathrm{i}S}\mathcal{D}\mu \tag{63}$$

where the action, which is complex, is given for continuous and differentiable paths by

$$\begin{aligned}S = \int_0^T \Bigg\{&\mathrm{i}[\gamma^*\dot{\alpha} + \delta^*\dot{\beta} + \xi^*\dot{\eta} + \rho^*\dot{\rho} + \sigma^*\dot{\sigma}] + \tfrac{1}{4}[(\alpha - \gamma^*)^2 + (\beta - \delta^*)^2] \\ &- \tfrac{1}{16}[(\alpha + \gamma^*)^2 + (\beta + \delta^*)^2]^2 - \mathrm{i}\left[\frac{(\xi^* + \eta)}{\sqrt{2}} + \frac{(\rho^* + \rho)}{\sqrt{2}}\right](\gamma^*\beta - \delta^*\alpha) \\ &- \frac{(\sigma^* + \sigma)}{\sqrt{2}}\frac{(\eta - \xi^*)}{\sqrt{2}\mathrm{i}}\Bigg\}\,\mathrm{d}t\end{aligned} \tag{64}$$

with boundary conditions specified in (44). Extremizing S we obtain the following equations of motion; variation with respect to γ^* and α lead to

$$\begin{aligned}\dot\alpha &= \frac{(\alpha-\gamma^*)}{2\mathrm{i}} + \frac{1}{4\mathrm{i}}(\alpha+\gamma^*)[(\alpha+\gamma^*)^2+(\beta+\delta^*)^2] + \frac{\beta}{\sqrt2}[(\xi^*+\eta)+(\rho^*+\rho)]\\ \dot\gamma &= \frac{(\gamma-\alpha^*)}{2\mathrm{i}} + \frac{1}{4\mathrm{i}}(\gamma+\alpha^*)[(\gamma+\alpha^*)^2+(\delta+\beta^*)^2] + \frac{\delta}{\sqrt2}[(\xi+\eta^*)+(\rho^*+\rho)]\end{aligned} \tag{65}$$

variation with respect to δ^* and β leads to

$$\begin{aligned}\dot\beta &= \frac{(\beta-\delta^*)}{2\mathrm{i}} + \frac{1}{4\mathrm{i}}(\beta+\delta^*)[(\alpha+\gamma^*)^2+(\beta+\delta^*)^2] - \frac{\alpha}{\sqrt2}[(\xi^*+\eta)+(\rho^*+\rho)]\\ \dot\delta &= \frac{(\delta-\beta^*)}{2\mathrm{i}} + \frac{1}{4\mathrm{i}}(\delta+\beta^*)[(\gamma+\alpha^*)^2+(\delta+\beta^*)^2] - \frac{\gamma}{\sqrt2}[(\xi+\eta^*)+(\rho^*+\rho)]\end{aligned} \tag{66}$$

and finally variation with respect to ξ^* and η lead to

$$\dot\eta = \frac{(\gamma^*\beta-\delta^*\alpha)}{\sqrt2} + \frac{(\sigma^*+\sigma)}{2} \qquad \dot\xi = \frac{(\alpha^*\delta-\beta^*\gamma)}{\sqrt2} + \frac{(\sigma^*+\sigma)}{2}. \tag{67}$$

Next, consider the trajectories $\alpha(t)$ and $\gamma(t)$; using (65) and the boundary conditions in (44) we find that for these paths

$$\begin{aligned}&\alpha(0)=\gamma(0)=\alpha'\\ &\dot\alpha(0)=\dot\gamma(0)=\left\{\frac{(\alpha'-\alpha'^*)}{2\mathrm{i}} + \frac{(\alpha'+\alpha'^*)}{4\mathrm{i}}[(\alpha'+\alpha'^*)^2+(\beta'+\beta'^*)^2]\right.\\ &\qquad\qquad\left. + \frac{\beta'}{\sqrt2}[(\eta'^*+\eta')+(\rho'^*+\rho')]\right\}.\end{aligned} \tag{68}$$

They have identical initial conditions and hence evolve along the same paths, i.e. $\alpha(t) = \gamma(t)$. Similarly it can be easily verified that $\beta(t) = \delta(t)$ and $\eta(t) = \xi(t)$. So using the fact that $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$ and the definitions in (48), we can write the evolution equations (65)–(67) in terms of the variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8,)$ and obtain the following equations:

$$\begin{aligned}&\dot q_1 = p_1 + q_2(q_3+q_7) \qquad \dot q_2 = p_2 - q_1(q_3+q_7) \qquad \dot p_7 = (q_1p_2 - q_2p_1)\\ &\dot p_1 = -q_1(q_1^2+q_2^2) + p_2(q_3+q_7) \qquad \dot p_3 = (q_1p_2-q_2p_1) \qquad \dot q_8 = 0\\ &\dot p_8 = -p_3 \qquad \dot p_2 = -q_2(q_1^2+q_2^2) - p_1(q_3+q_7) \qquad \dot q_3 = 0 \qquad \dot q_7 = 0.\end{aligned} \tag{69}$$

These are exactly the equations one would get from the classical phase space action

$$\begin{aligned}S' = \int_0^T &\left\{[k_1\dot x_1 + k_2\dot x_2 + \pi\dot y + p_\lambda\dot\lambda + p_\zeta\dot\zeta]\right.\\ &\left. - \left[\frac{\boldsymbol{k}^2}{2} + \frac14(\boldsymbol{x}^2)^2 + y\boldsymbol{k}T\boldsymbol{x} + \lambda\boldsymbol{k}T\boldsymbol{x} + \zeta\pi\right]\right\}\mathrm{d}t\end{aligned} \tag{70}$$

if, as before, one makes the identification

$$(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8) \leftrightarrow (x_1, k_1, x_2, k_2, x_3, k_3, \lambda, p_\lambda, \zeta, p_\zeta). \tag{71}$$

Hence, we see again that the action obtained from the quantum propagator gives us the desired classical equations of motion.

Now, let us substitute $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ and the definitions (48) in (64). One can obtain the real and imaginary parts of the action evaluated on the classical trajectories. We obtain for the real part, up to a total derivative,

$$S_1 \to \int_0^T \{[p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 + p_7\dot{q}_7 + p_8\dot{q}_8] - \tfrac{1}{2}(p_1^2 + p_2^2) - \tfrac{1}{4}(q_1^2 + q_2^2)^2 \\ -(q_3 + q_7)(p_1q_2 - q_1p_2) - q_8p_3\}\,\mathrm{d}t \tag{72}$$

just the classical phase space action of equation (70). The imaginary part reduces to

$$S_2 \to \frac{1}{2}\int_0^T [q_1\dot{q}_1 + p_1\dot{p}_1 + q_2\dot{q}_2 + p_2\dot{p}_2 + q_3\dot{q}_3 + p_3\dot{p}_3 + q_7\dot{q}_7 \\ + p_7\dot{p}_7 + q_8\dot{q}_8 + p_8\dot{p}_8]\,\mathrm{d}t \tag{73}$$

a surface term. Once again, we find that the complex action obtained from the quantum propagator gives us the correct classical evolution equations and when evaluated on the classical trajectories its real part is exactly equal to the classical phase space action evaluated on the same paths. The imaginary part along these paths is a surface term.

5. The measure

We would now like to make the point that the procedure developed here for constructing the path integral for the propagator is merely a recipe for obtaining the correct measure. We begin by noting that for a system with three dynamical degrees of freedom the unit operator is given by

$$1 = \int \frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z_3}{\pi^3}|z_1, z_2, z_3\rangle\langle z_1, z_2, z_3|. \tag{74}$$

However, the unit operator can also be written as

$$1 = \left\{\int \frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z_3}{\pi^3}|z_1, z_2, z_3\rangle\langle z_1, z_2, z_3|\right\}\left\{\int \frac{\mathrm{d}^2z_4\,\mathrm{d}^2z_5\,\mathrm{d}^2z_6}{\pi^3}|z_4, z_5, z_6\rangle\langle z_4, z_5, z_6|\right\} \\ = \int \frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z_3}{\pi^3}\frac{\mathrm{d}^2z_4\,\mathrm{d}^2z_5\,\mathrm{d}^2z_6}{\pi^3}|z_1, z_2, z_3\rangle\langle z_4, z_5, z_6| \\ \times \exp[-\tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2) + z_1^*z_4 + z_2^*z_5 + z_3^*z_6] \tag{75}$$

an integral over bicoherent states. So for the Hamiltonian $\mathcal{H} = \frac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x}) + y\boldsymbol{p}T\boldsymbol{x} + u\boldsymbol{p}T\boldsymbol{x} + v\pi$ we are studying here, we could have written the unit operator as

$$1 = \int \frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,\mathrm{d}^2\eta\,\mathrm{d}^2\gamma\,\mathrm{d}^2\delta\,\mathrm{d}^2\xi\,\mathrm{d}^2\rho\,\mathrm{d}^2\sigma}{\pi^8}|\alpha, \beta, \eta, \rho, \sigma\rangle\langle\gamma, \delta, \xi, \rho, \sigma| \\ \times \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2) + \alpha^*\gamma + \beta^*\delta + \eta^*\xi]. \tag{76}$$

Comparing this to the projection operator in (26) for our constrained system, which we reproduce below for convenience,

$$P = \int \frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta\,\mathrm{d}^2\eta\,\mathrm{d}^2\gamma\,\mathrm{d}^2\delta\,\mathrm{d}^2\xi\,\mathrm{d}^2\rho\,\mathrm{d}^2\sigma}{\pi^8}\frac{\mathrm{d}\theta}{2\pi}|\alpha, \beta, \eta, \rho, \sigma\rangle\langle\gamma, \delta, \xi, \rho, \sigma| \\ \times \exp\left[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2) \right. \\ \left. + \frac{(\alpha^{*2} + \beta^{*2})}{2}\,\mathrm{e}^{\mathrm{i}\theta} + \frac{(\gamma^2 + \delta^2)}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\right] \tag{77}$$

we see that the only difference between the projection operator and the unit operator is the measure over which the bicoherent states are integrated. Recall now the expression we obtained for the propagator using this projection operator at every time slice:

$$\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\, e^{-iT\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle \\ = \int \exp\left[i\int_0^T \{ i(\gamma^*\dot{\alpha} + \delta^*\dot{\beta} + \xi^*\dot{\eta} + \rho^*\dot{\rho} + \sigma^*\dot{\sigma}) - \langle\mathcal{H}\rangle\}\right]\mathcal{D}\mu \tag{78}$$

where the discrete form of the measure is given by

$$\mathcal{D}\mu = \prod_n \left\{ \frac{d^2\alpha_n\, d^2\beta_n\, d^2\eta_n\, d^2\gamma_n\, d^2\delta_n\, d^2\xi_n\, d^2\rho_n\, d^2\sigma_n}{\pi^8}\frac{d\theta_n}{2\pi} \right. \\ \times \exp\left[-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2) + \frac{(\alpha_n^{*2} + \beta_n^{*2})}{2}\, e^{i\theta_n} \right. \\ \left.\left. + \frac{(\gamma_n^2 + \delta_n^2)}{2}\, e^{-i\theta_n} + \gamma_n^*\alpha_n + \delta_n^*\beta_n + \xi_n^*\eta_n \right]\right\}. \tag{79}$$

On the other hand had we used the resolution of unity as written in (76) instead, the expression for our propagator would have been

$$\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|\, e^{-iT\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle \\ = \int \exp\left[i\int_0^T \{ i(\gamma^*\dot{\alpha} + \delta^*\dot{\beta} + \xi^*\dot{\eta} + \rho^*\dot{\rho} + \sigma^*\dot{\sigma}) - \langle\mathcal{H}\rangle\}\right]\mathcal{D}\mu_{\text{unit}} \tag{80}$$

and the measure would be

$$\mathcal{D}\mu_{\text{unit}} = \prod_n \left\{ \frac{d^2\alpha_n\, d^2\beta_n\, d^2\eta_n\, d^2\gamma_n\, d^2\delta_n\, d^2\xi_n\, d^2\rho_n\, d^2\sigma_n}{\pi^8} \right. \\ \times \exp[-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2) \\ \left. + \alpha_n^*\gamma_n + \beta_n^*\delta_n + \eta_n^*\xi_n + \gamma_n^*\alpha_n + \delta_n^*\beta_n + \xi_n^*\eta_n] \right\}. \tag{81}$$

So we see that both procedures would have led to the same action, over continuous and differentiable paths, but with quite different measures! Since the actions are the same they would yield identical classical equations of motion, but the different measures would give different spectrums to the quantization, only one of them being correct, of course.

It must be noted that a general operator admits a bicoherent state representation according to

$$O = \int |\alpha\rangle\langle\alpha|O|\beta\rangle\langle\beta| \frac{d^2\alpha\, d^2\beta}{\pi^2}. \tag{82}$$

Such operators have been dealt with, for example, by Glauber [12]. What is novel in the present paper is the use of such representations in path-integral constructions for which, in all previous coherent-state applications, only weighted coherent-state projection operators have been used.

6. Discussion and conclusions

The formalism surrounding the general theory of coherent states is exceptionally rich. In this paper, we have constructed the projection operator, equation (26), for a system with two first-class constraints, the projection operator being written as a properly-weighted integral over independent bras and kets, which were called *bicoherent states*. As the quantum states for the system evolve in time, recall equation (27), the projection operator projects these states at each infinitesimal time step of their evolution onto the physical subspace determined by the constraints on the system. The use of this projection operator leads to the correct measure for the path-integral representation of the propagator as discussed in section 5. The action obtained from a path-integral construction of the propagator using bicoherent states, in general, is complex. Also, the measure obtained from such a construction is path dependent and so the imaginary part of the action in effect modulates the measure.

The procedure outlined here has the additional desirable feature, whereby one can turn on and turn off the constraints as needed. For instance, if we wish the system to evolve under the constrained Hamiltonian $\mathcal{H}_1 = \frac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x}) + y\boldsymbol{p}T\boldsymbol{x} + u\boldsymbol{p}T\boldsymbol{x} + v\pi$ for an interval of time T_1 with the two first-class constraints considered in this paper and subsequently to evolve under an unconstrained Hamiltonian, say, $\mathcal{H}_2 = \frac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{x})$ for a period T_2 the propagator is given by

$$\begin{aligned}\langle\alpha'', \beta''; (T_1+T_2)|\alpha, \beta; 0\rangle = \int \frac{\mathrm{d}^2\alpha'\,\mathrm{d}^2\beta'}{\pi^2}\langle\alpha'', \beta''|\,\mathrm{e}^{-\mathrm{i}T_2\mathcal{H}_2}|\alpha', \beta'\rangle\delta(\alpha'^*\beta' - \beta'^*\alpha') \\ \times\left\langle\alpha', \beta', \frac{c_1T + c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}}\right| \\ \times\mathrm{e}^{-\mathrm{i}T_1\mathcal{H}_1}\left|\alpha, \beta, \frac{c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}}\right\rangle \end{aligned} \qquad (83)$$

where the factor $\langle\alpha'', \beta''|\,\mathrm{e}^{-\mathrm{i}T_2\mathcal{H}_2}|\alpha', \beta'\rangle$ in the integrand above is evaluated in the usual manner by introducing resolutions of unity at each time slice, whereas the term

$$\left\langle\alpha', \beta', \frac{c_1T + c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}}\right|\mathrm{e}^{-\mathrm{i}T_1\mathcal{H}_1}\left|\alpha, \beta, \frac{c_5}{\sqrt{2}}, \frac{c_3 + \mathrm{i}c_4}{\sqrt{2}}, \frac{c_1 + \mathrm{i}c_2}{\sqrt{2}}\right\rangle$$

would be evaluated as discussed in section 3. The Dirac delta function in the integrand in (83) ensures that at time $t = T_1$ the system is on the constraint surface.

The 'quantization recipe' developed here and applied to a system with first-class constraints will be extended to systems with second-class constraints in a forthcoming paper. Also, our formalism may possibly be extended beyond the simple models considered here.

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Appendix

In this section we highlight the main features of path integrals constructed using bicoherent states. For simplicity we consider a system without constraints and with a single degree of freedom. To begin with, notice that the unit operator can be written as

$$\begin{aligned} 1 = 1^2 &= \int \frac{\mathrm{d}^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \int \frac{\mathrm{d}^2\beta}{\pi}|\beta\rangle\langle\beta| \\ &= \int \frac{\mathrm{d}^2\alpha\,\mathrm{d}^2\beta}{\pi^2}|\alpha\rangle\langle\beta| \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta] \end{aligned} \qquad (84)$$

a weighted integral over bicoherent states. We will use this form of unity in our construction of the path-integral representation of the propagator. Consider the Hamiltonian $H = p^2/2 + V(x)$. In the construction of the propagator we use the normal-ordered Hamiltonian $\mathcal{H} =: H :$ when convenient. Thus, the propagator is given by

$$\begin{aligned}\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha'\rangle &= \int\ldots\int\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_N\rangle\langle\beta_N|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_{N-1}\rangle\ldots\langle\beta_1|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha'\rangle\prod_{n=1}^{N}\mathrm{d}\mu_n''\\ &= \int\ldots\int\prod_{n=0}^{N}\langle\beta_{n+1}|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_n\rangle\prod_{n=1}^{N}\mathrm{d}\mu_n''\end{aligned}\tag{85}$$

where $(N+1)\epsilon = T$ and $n = 1, 2, \ldots, N$. In the equation above, the boundary conditions are $(\alpha_0, \beta_{N+1}) = (\alpha', \alpha'')$ and the measure at each time slice is given by

$$\mathrm{d}\mu_n'' = \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n}{\pi^2}\exp[-\tfrac{1}{2}(|\alpha_n|^2 + |\beta_n|^2) + \alpha_n^*\beta_n].\tag{86}$$

For small ϵ, we evaluate, to order ϵ, each term in the integrand in (85) as follows:

$$\langle\beta_{n+1}|\,\mathrm{e}^{-\mathrm{i}\epsilon\mathcal{H}}|\alpha_n\rangle \simeq \langle\beta_{n+1}|[1-\mathrm{i}\epsilon\mathcal{H}]|\alpha_n\rangle = \langle\beta_{n+1}|\alpha_n\rangle[1-\mathrm{i}\epsilon H_{n+1,n}] \simeq \langle\beta_{n+1}|\alpha_n\rangle\,\mathrm{e}^{-\mathrm{i}\epsilon H_{n+1,n}}\tag{87}$$

where $H_{n+1,n}$ in the expression above is

$$H_{n+1,n} = \frac{\langle\beta_{n+1}|\mathcal{H}|\alpha_n\rangle}{\langle\beta_{n+1}|\alpha_n\rangle}.\tag{88}$$

Also, the overlap of coherent states at each time slice is

$$\langle\beta_{n+1}|\alpha_n\rangle = \exp[-\tfrac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2) + \beta_{n+1}^*\alpha_n].\tag{89}$$

Thus, provided the integrals exist, the propagator is

$$\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha'\rangle = \int\ldots\int\prod_{n=0}^{N}\exp[-\tfrac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2) + \beta_{n+1}^*\alpha_n - \mathrm{i}\epsilon H_{n+1,n}]\prod_{n=1}^{N}\mathrm{d}\mu_n''.\tag{90}$$

We notice that the factor $\exp[-\frac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2)]$, except at the endpoints, can be absorbed in the measure. So, our propagator becomes

$$\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha'\rangle = \exp[-\tfrac{1}{2}(|\alpha''|^2 + |\alpha'|^2)]\int\ldots\int\exp\left[\sum_{n=0}^{N}(\beta_{n+1}^*\alpha_n - \mathrm{i}\epsilon H_{n+1,n})\right]\prod_{n=1}^{N}\mathrm{d}\mu_n'\tag{91}$$

where the measure at each time slice has changed slightly and is now given by

$$\mathrm{d}\mu_n' = \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n}{\pi^2}\exp[-(|\alpha_n|^2 + |\beta_n|^2) + \alpha_n^*\beta_n].\tag{92}$$

In our quest to express the right-hand side of (91) as a path integral, we rewrite part of the exponent as follows:

$$\sum_{n=0}^{N}\beta_{n+1}^*\alpha_n = \sum_{n=0}^{N}\tfrac{1}{2}[(\beta_{n+1}^* - \beta_n^*)\alpha_n - \beta_{n+1}^*(\alpha_{n+1} - \alpha_n)] + \sum_{n=0}^{N}\tfrac{1}{2}[\beta_n^*\alpha_n + \beta_{n+1}^*\alpha_{n+1}].\tag{93}$$

Notice that in the above equation the terms (β_0, α_{N+1}) have not been defined yet. The factors containing these terms cancel and so they can take on arbitrary values. We assign them the values $(\beta_0, \alpha_{N+1}) = (\alpha', \alpha'')$. Also note the second sum in (93) can be absorbed in the measure and so our propagator is finally written as

$$\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha'\rangle = \int \ldots \int \exp\left[\sum_{n=0}^{N}\{\tfrac{1}{2}[(\beta^*_{n+1} - \beta^*_n)\alpha_n - \beta^*_{n+1}(\alpha_{n+1} - \alpha_n)] - \mathrm{i}\epsilon H_{n_1,n}\}\right] \times \prod_{n=1}^{N} \mathrm{d}\mu_n \tag{94}$$

where the measure at each slice is now

$$\mathrm{d}\mu_n = \frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n}{\pi^2}\exp[-(|\alpha_n|^2 + |\beta_n|^2) + \alpha^*_n\beta_n + \beta^*_n\alpha_n]. \tag{95}$$

Thus, in the limit $N \to \infty$, $\epsilon \to 0$ with $(N+1)\epsilon = T$, the right-hand side of (94) can be formally written as an integral over continuous and differentiable paths and so our propagator is given by

$$\langle\alpha''|\,\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\alpha'\rangle = \int \exp\left[\mathrm{i}\int\{\tfrac{\mathrm{i}}{2}(\beta^*\dot{\alpha} - \dot{\beta}^*\alpha) - \langle\mathcal{H}\rangle\}\,\mathrm{d}t\right]\mathcal{D}\mu \tag{96}$$

where

$$\langle\mathcal{H}\rangle = \frac{\langle\beta|\mathcal{H}|\alpha\rangle}{\langle\beta|\alpha\rangle} \tag{97}$$

and the measure is

$$\mathcal{D}\mu = \prod_n \left\{\frac{\mathrm{d}^2\alpha_n\,\mathrm{d}^2\beta_n}{\pi^2}\exp[-(|\alpha_n|^2 + |\beta_n|^2) + \alpha^*_n\beta_n + \beta^*_n\alpha_n]\right\}. \tag{98}$$

Notice that the phase space action

$$S = \int\left\{\frac{\mathrm{i}}{2}(\beta^*\dot{\alpha} - \dot{\beta}^*\alpha) - \langle\mathcal{H}\rangle\right\}\mathrm{d}t \tag{99}$$

obtained from the path-integral representation of the propagator constructed using bicoherent states is complex.

In the case of the harmonic oscillator, for which $\mathcal{H} = a^\dagger a$, the action is

$$S = \int\left\{\frac{\mathrm{i}}{2}(\beta^*\dot{\alpha} - \dot{\beta}^*\alpha) - \beta^*\alpha\right\}\mathrm{d}t. \tag{100}$$

Defining $\alpha = (q_1 + \mathrm{i}p_1)/\sqrt{2}$, $\beta = (q_2 + \mathrm{i}p_2)/\sqrt{2}$, and inserting these definitions in the action above we obtain

$$S = \int\{\tfrac{1}{4}(p_2\dot{q}_1 - q_2\dot{p}_1 + p_1\dot{q}_2 - q_1\dot{p}_2) - \tfrac{1}{2}(q_2q_1 + p_2p_1)\}\,\mathrm{d}t + \mathrm{i}\int\{\tfrac{1}{4}(q_2\dot{q}_1 + p_2\dot{p}_1 - q_1\dot{q}_2 - p_1\dot{p}_2) - (q_2p_1 - p_2q_1)\}\,\mathrm{d}t. \tag{101}$$

In this paper, we identify the real part of the action obtained from the bicoherent state construction, with the standard classical phase space action. We see that for the example of the harmonic oscillator the action obtained from our construction using bicoherent states has twice as many labels as the usual action. The doubling of labels and the fact that the action is complex are two general features of path integrals constructed using bicoherent states.

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